Exact discrete compactlike traveling kinks and pulses in ϕ^4 nonlinear lattices

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We show that by properly choosing the analytical form of a solitary wave solution of discrete ϕ^4 models we can calculate the parameters of the potential which allow the propagation of compact (kink and pulses) solutions. Our numerical simulations show that narrow kinks and pulses with finite extent can propagate freely, and that discrete breathers with finite but long lifetime, can emerge from their collisions. Moreover, our numerical simulations reveal that the propagation of two successive pulses at a relative distance of two lattice spacings propagate freely, i.e., without interaction.

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I. INTRODUCTION

In recent years, the dynamics of kinks in Hamiltonian (nondissipative or Klein-Gordon) systems [1] has attracted considerable attention. It becomes clear that continuous propagation equations with linear coupling provide an inadequate description of the behavior of weakly coupled lattices where the interplay between nonlinearity spatial discreteness can lead to effects not present in the continuum models. For instance, lattices such as ferromagnetic chains [2], hydrogen bonded chains [3], or chains of base pairs in DNA [4], kink solitons, or domain walls, whose width is of the order of a few lattice spacings, may be pin in the lattice owing to discreteness effects. On the other hand, classical equations which describe the behavior of the previously cited systems possess extended spatial solutions that may be incorrect from a physical point of view.

In order to gain an understanding of wave motion in discrete systems, where exact results are scarce even in one dimension both for linear and nonlinear interaction, it is desirable to investigate lattice models with exact solutions. In this direction, Schmidt [5] pointed out that if a double-well on-site potential of a ϕ^4 model is suitably chosen, the single kink soliton becomes an exact solution of the discrete model. Recently, [6,7] the general problem was considered of finding kink- or pulse-shaped traveling-waves solutions separately in the conservative and dissipative case with a linear interaction coupling, giving place to infinite extent solution (tanh shaped). However, observed patterns in nature whether stationary or traveling are of finite extent. Indeed, it was recently shown by Rosenau and Hyman [12–14], that solitary-wave solutions may compactify under the influence of nonlinear dispersion which is capable of causing deep qualitative changes in the nature of nonlinear phenomena. Such robust solitonlike solutions, characterized by the absence of an infinite tail or wings and whose width is velocity independent, have been called compactons [14–16]. One might therefore wonder if it is possible to construct a discrete model including nonlinear coupling, allowing the propagation of compactlike wave fronts and pulses.

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The purpose of this paper is to make some progress in the understanding of the effects of discreteness and nonlinear interactions on the dynamical behavior of one-dimensional nonlinear ϕ^4 lattices.

The paper is organized as follows. First, we present our specific lattice model and show analytically that it can admit exact compactlike kink solutions if parameters of the ϕ^4 potential is adequately chosen. Then, in Sec. III, we study numerically the propagation and the collisions of such compactlike kinks and antikinks, in the discrete case and in the continuous limit. In Sec. IV, we show that compactlike pulses may be solutions of our system. In Sec. V, we study their propagation and their collision. Finally, Sec. VI is devoted to concluding remarks.

II. MODEL AND EQUATION OF MOTION

We consider a lattice model where a system of atoms with unit mass are coupled anharmonically to their nearest neighbors and interact with a nonlinear substrate potential $V(u_n)$. The Hamiltonian of the system is given by

$$H=\sum_n \left(\frac{1}{2}\dot{u}_n^2+\frac{K}{(\gamma+1)}(u_{n+1}-u_n)^{\gamma+1}+V(u_n)\right), \quad (1)$$

where u_n is the scalar dimensionless displacement of the nth atom. Constants K and γ are, respectively, the stiffness coupling term and the strength interaction law between nearest-neighbors atoms. In the specific case $\gamma=3$ (which will be considered in this paper), the corresponding equation of motion of the nth atom is

$$\ddot{u}_n=K[(u_{n+1}-u_n)^3-(u_n-u_{n-1})^3]-\frac{dV(u_n)}{du_n}. \quad (2)$$

Let us introduce $u_n=u_0\ \psi_n$, where u_0 is a constant, and setting $\omega_0^2=K\ u_0^2$, and $\Omega_0^2=1/u_0$, Eq. (2) becomes

$$\ddot{\psi}_n=\omega_0^2[(\psi_{n+1}-\psi_n)^3-(\psi_n-\psi_{n-1})^3]-\Omega_0^2F(\psi_n). \quad (3)$$

Finally, dividing the two members of Eq. (3) by ω_0^2, and setting $t'=\omega_0 t$ (dimensionless time), and $\Gamma=\Omega_0^2/\omega_0^2$, we get

$$\frac{d^2\psi_n}{dt^2}=[(\psi_{n+1}-\psi_n)^3-(\psi_n-\psi_{n-1})^3]-\Gamma F(\psi_n). \quad (4)$$

Assuming that, a traveling compactlike kink or compacton solution has the following kink shape:

$$\psi_n=\sin(s),\quad \text{if } s\in[-\pi/2,+\pi/2],$$
$$\psi_n=-1,\quad \text{if } s\in]-\infty,-\pi/2[, \quad (5)$$
$$\psi_n=+1,\quad \text{if } s\in]+\pi/2,+\infty[,$$

where $s=\omega t'-kna$. Here, a is the lattice spacing, and ω and k are two constants such that the ratio ω/k represents the velocity of the front wave. Contrary to the linear coupling models proposed [5–7] for the description of the dynamic behavior of such systems, where the tanh-shaped wave-front solution extends asymptotically to infinity, solution (5) has the advantage to taking into account the finite spatial extent of a physical or real wave front. Now, following an inverse procedure, we first insert Eq. (5) in Eq. (3), in order to calculate the expression of $F(\psi_n)$. Thus,

$$\frac{d^2\psi_n}{dt^2}=-\omega^2\psi_n. \quad (6)$$

$$(\psi_{n+1}-\psi_n)^3=[\sin(s-\xi)-\sin(s)]^3 \times(\sin s\cos\xi+\sin\xi\cos s-\sin s)^3, \quad (7)$$

$$(\psi_n-\psi_{n-1})^3=[\sin(s)-\sin(s+\xi)]^3 \times(\sin s-\sin s\cos\xi-\sin\xi\cos s)^3, \quad (8)$$

with $\xi=ka$. Setting $A=-\sin\xi\cos s$ and $B=\sin s(\cos\xi-1)$, the cubic difference becomes

$$\Delta=(\psi_{n+1}-\psi_n)^3-(\psi_n-\psi_{n-1})^3=2B(B^2+3A^2). \quad (9)$$

That is,

$$\Delta=2(\tau-1)^2\psi_n[4(\tau+1/2)\psi_n^2-3(1+\tau)], \quad (10)$$

with $\tau=\cos(\xi)$. Using the previous expressions we deduce that the substrate force is

$$F(\psi_n)=\frac{1}{\Gamma}\left(\alpha\psi_n+\beta\psi_n^3\right), \quad (11)$$

with

$$\alpha=\omega^2-6(\tau+1)(\tau-1)^2, \text{ and}$$
$$\beta=8(\tau+1/2)(\tau-1)^2. \quad (12)$$

Finally the total equation becomes

$$\frac{d^2\psi_n}{dt^2}=[(\psi_{n+1}-\psi_n)^3-(\psi_n-\psi_{n-1})^3]-\beta\psi_n\left(\psi_n^2+\frac{\alpha}{\beta}\right). \quad (13)$$

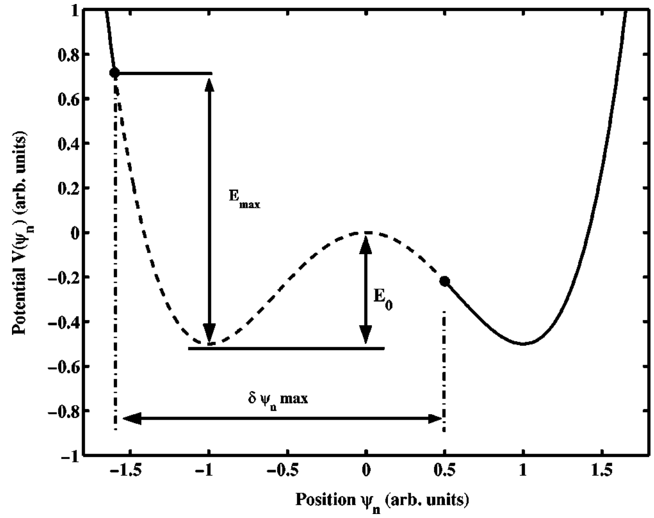

FIG. 1. Symmetric potential $V(\psi_n)$, with its two degenerated minima. The shape is obtained with the parameters $\tau=\cos(\pi/3)$ and $\omega=0.5$. The dashed line represents the asymmetric oscillations of the central particles of the breather (with an amplitude $\delta\psi_n$ max $=2.1$) created by the kink-antikink collision. $E_{\max}$ represents the maximum of energy of the particles during the oscillations while $E_0=0.5$ is the barrier height.

This last equation presents a ϕ^4 substrate potential structure given by Eq. (14) with two degenerated minima if the ratio α/β is negative and $\beta>0$ (see Fig. 1).

$$V(\psi_n)=\beta\left(\frac{\psi_n^4}{4}+\frac{\alpha}{\beta}\frac{\psi_n^2}{2}\right). \quad (14)$$

Since we have assumed that the solution of Eq. (13) has the form (5), the two minima must be located, respectively, at $\psi_n=-1$ and $\psi_n=+1$, which corresponds to a ratio $\alpha/\beta=-1$. This, gives us the existence condition of our compact kink solution, and then define the value of parameter ω, for a fixed constant discrete parameter τ or reciprocally, which define a bound for $\xi=ka$. Therefore, the wave-front velocity is then given by

$$V_\phi=\frac{\sqrt{2(1-\tau)^3}}{\arccos(\tau)}=\frac{\sqrt{2[1-\cos(ka)]^3}}{ka}=\frac{\sqrt{2[1-\cos(\xi)]^3}}{\xi}. \quad (15)$$

We note that, the velocity of the wave front is associated to discrete parameter ka.

III. NUMERICAL RESULTS: FRONT DYNAMICS

A. Compactlike kink propagation

We have checked by numerical simulations that an exact discrete compact kink solution given by Eq. (5) can propagate freely, that is without experiencing any discreteness effects for the parameter range $ka\leqslant\pi/3$. From this value on, some discreteness effects appear owing to the fact that it is very difficult to describe a sine function only with three points or two lattice spacing [see Fig. 2(a)]. Figures 2(a) and

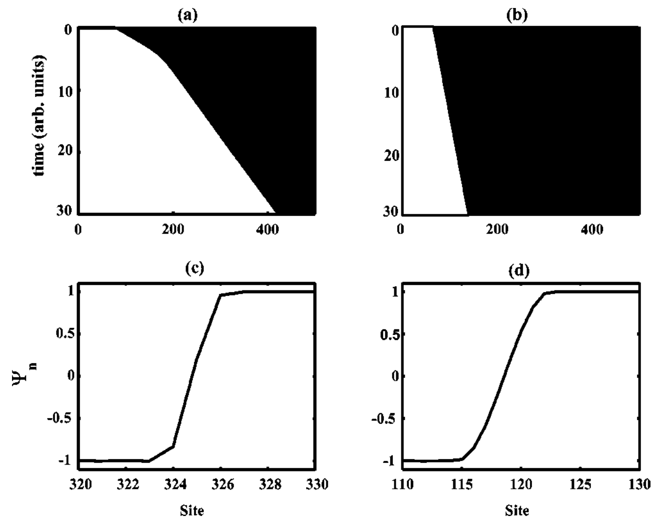

FIG. 2. Compact kinks propagation and shape: (a) Spatiotemporal evolution for a solution with discrete parameter $ka=\pi/2$. (b) Spatiotemporal evolution for a solution with discrete parameter $ka=\pi/8$. (c) Structure of the solution after a time $t'=20$ (expressed in arbitrary unit) for discrete parameter $ka=\pi/2$. (d) Structure of the solution after a time $t'=20$ (expressed in arbitrary unit) for discrete parameter $ka=\pi/8$.

2(b) show the spatiotemporal dynamics of compactlike kinks, respectively, for $ka=\pi/2$ and $ka=\pi/8$.

One can remark [Fig. 2(a) (discrete parameter $ka=\pi/2$)] that kink velocity decreases strongly with time until reaching a speed limit corresponding to a solution which is different to the predicted one. This means that there exists other exact compact solutions in such a system. Indeed, as shown in Fig. 2(c), the profile of this numerical solution is also a compactlike solution of Eq. (13).

Unlike the very discrete case discussed previously, kink compacton with discrete parameter $ka=\pi/8$ propagates freely, i.e., without emission of radiation, thus confirming the exact character of solution (5). Figure 2(d) shows the profile solution after a time $t'=20$ (expressed in arbitrary units). This profile is identical to high accuracy to the initial condition. A systematic investigation of the velocity and the emission of radiation reveals that the critical value of parameter ka over which a solution of type (5) radiates and consequently reduces its velocity is $ka=\pi/3$.

Note that this value of discrete parameter ($ka=\pi/3$) already corresponds to a very discrete situation, since the construction of the corresponding solution requires only three lattice spacings.

B. Breather generation

We have also studied the possible generation of nonlinear localized modes or compactlike breathers via kink (K) and antikink ($\bar{K}$) collisions, respectively, in the discrete and continuous regime. This interesting type of nontopological excitation appears in a large variety of nonlinear lattices and their existence related to energy localization [8–11]. Here, we studied numerically a K-$\bar{K}$ collision in the discrete ($ka=\pi/3$) and the continuous limit ($ka=\pi/8$) cases, respectively. First, in the discrete case the two entities travel with velocity $v_\phi \simeq 0.478\ \text{cells}^{-1}$ and $-v_\phi$, respectively. As shown in Fig. 3(a), a discrete stationary breather and small amplitude nonlinear oscillations background emerge from the weakly inelastic K-$\bar{K}$ collision. The asymmetric breather amplitude oscillations (see Fig. 1) decrease slowly with time [see Fig. 3(c)]. Although the breather appears to be stable, after time $t=60$ (a.u.), the breather sink to chaotic oscillations of small amplitude. Nevertheless, in spite of these weak nonlinear radiation losses (which propagate away), this discrete breather has a significant lifetime and presents a genuine physical interest. Note that the interaction between these two compact entities is very different compared to the case of tanh-shaped or spatially extended solutions which interact at long distances. Indeed, their collision may be compared to that of two hard spheres, i.e., without long distance interaction.

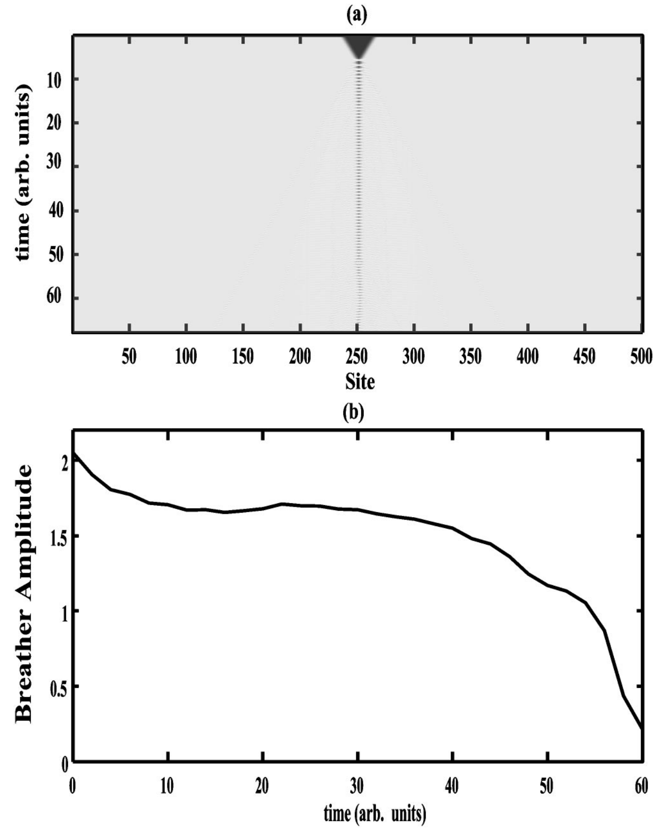

FIG. 3. (a) Creation of a discrete stationary breather with nonlinear oscillation background generated by a $K-\bar{K}$ collision. The oscillations of the central particle are asymmetric with an amplitude $\delta\psi_n$ max$=2.1$ (see Fig. 1) and a period $T_B=0.7$ (a.u.). (b) Breather amplitude decreasing versus time (in arbitrary unit).

IV. COMPACTLIKE PULSE SOLUTIONS

As seen in Sec. II, solutions of Eq. (13) are strictly localized sine functions $\psi_n=\pm\sin(s)$ defined on interval $s\in[-\pi/2,+\pi/2]$, and ±1 otherwise [see Fig. 2(d)]. Taking into account the symmetry of these previous solutions, a straightforward calculation shows that

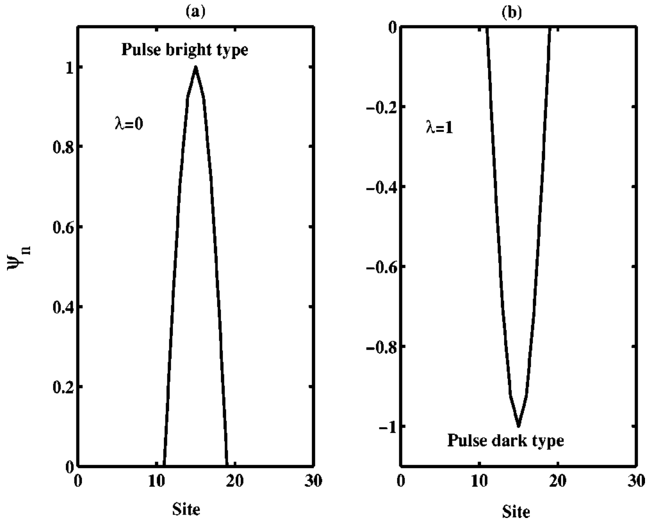

FIG. 4. Pulse shape solutions of Eq. (13) Left: a bright type compactlike pulse, corresponding to $\lambda=0$ in Eq. (16). Right: a dark type compactlike pulse, corresponding to $\lambda=1$ in Eq. (16). Both solutions have discrete parameter $ka=\pi/8$.

$$\psi_n=(-1)^{\lambda}\cos(s),\quad \text{if } s\in[-\pi,+\pi],$$

$$\psi_n=(-1)^{\lambda+1},\quad \text{otherwise,} \tag{16}$$

are also solutions of Eq. (13). The parameter λ is an integer equal to zero or one. If $\lambda=0$ the solution corresponds to a bright type compactlike pulse (see Fig. 4(a)), and if $\lambda=1$ the solution corresponds to a dark type compactlike pulse [see Fig. 4(b)]. Note that the pulses velocity is also given by Eq. (15). Therefore, in the following, each pulse will be associated with discrete parameter *ka*.

V. NUMERICAL RESULTS: PULSE DYNAMICS

A. Compactlike pulse propagation

We have checked numerically the stability of the solutions given by Eq. (16) and their exact discrete compact character for discrete parameter $ka\leqslant\pi/3$. As shown in Figs. 5(a) and 5(b) the solutions propagate freely, that is without emission of radiation or nonlinear oscillations. Likewise with the front dynamics, a systematic investigation of emission of radiation

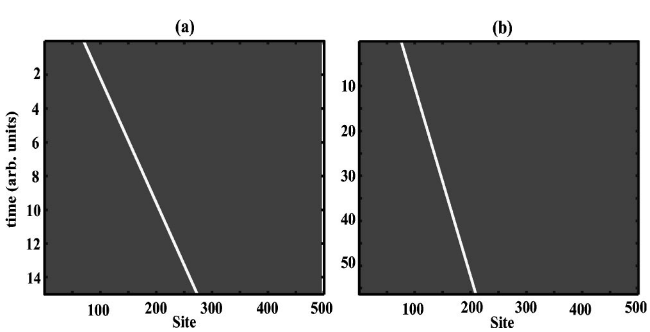

FIG. 5. (a) Pulse propagation for discrete parameter $ka=\pi/3$ (width: six lattice spacings). (b) Pulse propagation for discrete parameter $ka=\pi/8$ (width: eight lattice spacings).

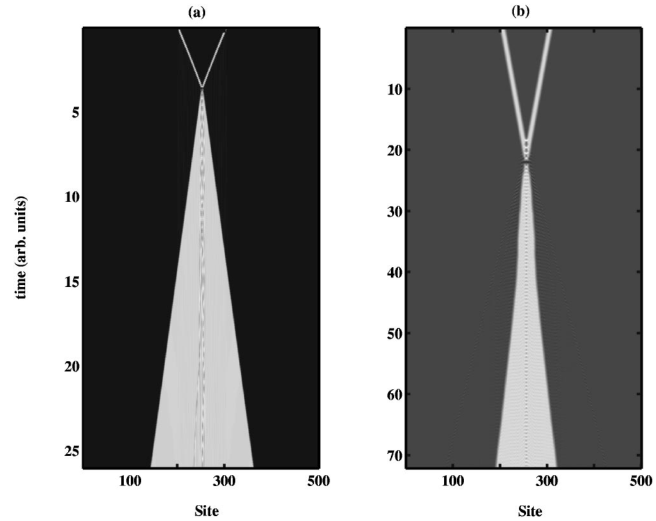

FIG. 6. Compactlike collision. (a) Collision between two pulses in discrete regime: discrete parameter $ka=\pi/3$. (b) Collision between two pulses in the continuous limit: discrete parameter $ka=\pi/8$.

and velocity, reveals that the critical value of the discrete parameter *ka* over which a solution of type (16) radiates or emits nonlinear oscillations is $ka\leqslant\pi/3$. Note that, from a numerical point of view, it is very important to describe correctly the horizontal asymptote on the pulse top (when $ka\simeq\pi/3$). Indeed, nonlinear oscillations occur if this condition is not respected. The specific case (not shown here), corresponding to a discrete parameter $ka=\pi/2$, leads to an unstable pulse, that tends to widen along the propagation until it reaches a stable width. This pulse is then composed of two complementary fronts (K-$\bar{K}$) as seen in Sec. III [see Fig. 2(c)] and traveling in the same direction. The final width after widening is equal to 17 lattice spacings. This effect stems from the fact that it is very difficult to define a sine function and its horizontal asymptote with only three points or two lattice spacings. Therefore, to obtain and propagate compactlike pulses in a such system, it is necessary to define the solution on six lattice spacings at least, as well as define correctly its horizontal asymptote.

B. Compactlike pulse collisions

As in Sec. III, we have investigated the collisions between two pulses in the discrete regime and in the continuous limit. As shown in Fig. 6, in the two cases, the collision leads to a localized mode, and to counter propagating compactlike kinks [see Figs. 6(a) and 6(b)]. In the discrete regime this mode is unstable [see Fig. 6(a)]. Indeed, its interaction with the nonlinear oscillation background splits it in two modes which move erratically and disappear in small amplitude chaotic oscillations.

On the other hand, in the continuous limit [see Fig. 6(b)], the generated mode is very stable in amplitude, time, and position, and therefore presents a physical interest. Moreover, we have verified that two adjacent pulses spaced of two lattice spacing can propagate freely, that is without any interaction. A systematic study reveals that the discrete parameter must be lower than $\pi/3$. Indeed, for values higher than this critical value, any interactions occur between the two successive pulses because of the nonlinear oscillation background that is generated during the propagation by the exceedingly discrete solutions.

VI. CONCLUDING REMARKS

We have explored the dynamics of a ϕ^4 lattice model with nonlinear coupling interaction between nearest-neighbor atoms. We have first shown that by properly choosing the analytical form of a discrete solitary wave or compactlike solution of the model we can calculate analytically the parameters of the ϕ^4 potential. The compactlike kink solutions are sine shaped and therefore strictly localized, that is without wings or tails. We have checked numerically that discrete compactlike kink (antikink) solutions can propagate freely without experiencing any discreteness effects if the discrete parameter ka is lower or equal to $\pi/3$. Kink-antikink collisions reveal that static breather with finite but with physically interesting lifetimes can be generated. Moreover, we have shown that compactlike pulses can also be solutions of such systems, and also propagate freely. Their collisions are pseudoelastic and give birth to localized modes which are unstable in the discrete regime, but very stable in the continuous limit, and consequently are able to play a role in physical processes. We have also studied, but not presented, the propagation of two consecutive pulses spaced of two lattice spacing and seen that no interaction between them occurs if the discrete parameter ka is smaller than $\pi/3$. We would like to point out again that our model and results are relevant for physical systems in which lattice discreteness is important. Obviously, further studies are necessary especially by including a linear coupling and a dissipative term, to determine all the properties of these compactlike kinks and pulses with exceptional mobilities. In conclusion, we believe that the understanding of discrete nonlinear models is an active and attractive topic of the current research. Since realistic physical systems are rather complicated, it is extremely important to develop the basic concepts with help of simple lattice models with exact solutions.

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